On sg-Separation Axioms

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ABSTRACT

In this paper we define almost sg-normality and mild sg-normality, continue the study of further properties of sg-normality. We show that these three axioms are regular open hereditary. Also define the class of almost sg-irresolute mappings and show that sg-normality is invariant under almost sg-irresolute M-sg-open continuous surjection.

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1. INTRODUCTION

In 1967, A. Wilansky has introduced the concept of US spaces. In 1968, C.E. Aull studied some separation axioms between the T₁ and T₂ spaces, namely, S₁ and S₂. Next, in 1982, S.P. Arya et al have introduced and studied the concept of semi-US spaces and also they made study of s-convergence, sequentially semi-closed sets, semi-compact notions. G.B. Navlagi studied P-Normal Almost-P-Normal, Mildly-P-Normal and Pre-US spaces. Recently S. Balasubramanian and P. Aruna Swathi Vijayanthi studied v-Normal Almost- v-Normal, Mildly-v-Normal and v-US spaces. Inspired with these we introduce sg-Normal Almost- sg-Normal, Midly- sg-Normal, sg-US, sg-S₁ and sg-S₂. Also we examine sg-convergence, sequentially sg-compact, sequentially sg-continuous maps, and sequentially sub sg-continuous maps in the context of these new concepts. All notions and symbols which are not defined in this paper may be found in the appropriate references. Throughout the paper X and Y denote Topological spaces on which no separation axioms are assumed explicitly stated.

2. PRELIMINARIES

2.1. Definition 2.1

A ⊆ X is called (i) g-closed if cl(A) ⊆ U whenever A ⊆ U and U is open in X.
(ii) sg-closed if scl(A) ⊆ U whenever A ⊆ U and U is semiopen in X.

2.2. Definition 2.2

A space X is said to be
(i) T₁(T₂) if for any x ≠ y in X, there exist (disjoint) open sets U; V in X such that x ∈ U and y ∈ V.
(ii) Weakly Hausdorff if each point of X is the intersection of regular closed sets of X.
(iii) Normal [resp: mildly normal] if for any pair of disjoint [resp: regular-closed] closed sets F₁ and F₂ , there exist disjoint open sets U and V such that F₁ ⊆ U and F₂ ⊆ V.
(iv) Almost normal if for each closed set A and each regular closed set B such that A ∩ B = ∅, there exist disjoint open sets U and V such that A ⊆ U and B ⊆ V.
(v) Weakly regular if for each pair consisting of a regular closed set A and a point x such that A ∩ {x} = ∅, there exist disjoint open sets U and V such that x ∈ U and A ⊆ V.
(vi) A subset A of a space X is S-closed relative to X if every cover of A by semiopen sets of X has a finite subfamily whose closures cover A.
(vii) R₀ if for any point x and a closed set F with x ∉ F in X, there exists a open set G containing F but not x.
(viii) R₁ if for x, y ∈ X with cl(x) ≠ cl(y), there exist disjoint open sets U and V such that cl(x) ⊆ U, cl(y) ⊆ V.
(ix) US-space if every convergent sequence has exactly one limit point to which it converges.
(x) pre-US space if every pre-convergent sequence has exactly one limit point to which it converges.
(xi) pre-S₁ if it is pre-US and every sequence <xₙ> pre-converges with subsequence of <xₙ> pre-side points.
(xii) pre-S₂ if it is pre-US and every sequence <xₙ> in X pre-converges which has no pre-side point.
(xiii) is weakly countable compact if every infinite subset of X has a limit point in X.
(xiv) Baire space if for any countable collection of closed sets with empty interior in X, their union also has empty interior in X.

2.3. Definition 2.3

Let A ⊆ X. Then a point x is said to be a
(i) limit point of A if each open set containing x contains some point y of A such that x ≠ y.
(ii) T₀-limit point of A if each open set containing x contains some point y of A such that cl(x) = cl(y), or equivalently, such that they are topologically distinct.
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(iii) pre-$T_\infty$-limit point of $A$ if each open set containing $x$ contains some point $y$ of $A$ such that $pc(x) \neq pc(y)$, or equivalently, such that they are topologically distinct.

Note 1: Recall that two points are topologically distinguishable or distinct if there exists an open set containing one of the points but not the other; equivalently if they have disjoint closures. In fact, the $T_\infty$-axiom is precisely to ensure that any two distinct points are topologically distinct.

Example 1: Let $X = \{a, b, c, d\}$ and $\tau = \{(a, b, c), (a, b, X), (X, d)\}$. Then $b$ and $c$ are the limit points but not the $T_\infty$-limit points of the set $\{b, c\}$. Further $d$ is a $T_\infty$-limit point of $\{b, c\}$.

Example 2: Let $X = (0, 1)$ and $\tau = \{(q, X), \{a, b\}, (a, b, c, X)\}$. Then every point of $X$ is a limit point of $X$. Every point of $X-U_2$ is a $T_\infty$-limit point of $X$. But no point of $U_2$ is a $T_\infty$-limit point of $X$.

2.4. Definition 2.4

A set $A$ together with all its $T_\infty$-limit points will be denoted by $T_\infty$-$clA$.

Note 2: i. Every $T_\infty$-limit point of a set $A$ is a limit point of the set but the converse is not true in general.

ii. In $T_\infty$-space both are same.

Note 3: $R_0$-axiom is weaker than $T_\infty$-axiom. It is independent of the $T_0$-axiom. However $T_1 = R_0 + T_0$.

Note 4: Every countable compact space is weakly countable compact but converse is not true in general. However, a $T_1$-space is weakly countable compact if it is countable compact.

3. $sg-T_0$ LIMIT POINT

3.1. Definition 3.01
In $X$, a point $x$ is said to be a $sg-T_0$-limit point of $A$ if each $sg$-open set containing $x$ contains some point $y$ of $A$ such that $sgcl(x) = sgcl(y)$, or equivalently, such that they are topologically distinct with respect to $sg$-open sets.

Note 5: regular open set $\Rightarrow$ open set $\Rightarrow$ semi-open set $\Rightarrow$ open set we have $r-T_0$-limit point $\Rightarrow$ $T_0$-limit point $\Rightarrow$ $s-T_0$-limit point $\Rightarrow$ $sg-T_0$-limit point

Example 3: Let $X = \{a, b, c\}$ and $\tau = \{(a, b, c), (a, b, X), (X, d)\}$. For $A = \{a, b\}$, $a$ is $sg-T_0$-limit point.

3.2. Definition 3.02

A set $A$ together with all its $sg-T_0$-limit points is denoted by $T_0$-$sgcl(A)$

3.3. Lemma 3.01

If $x$ is a $sg-T_0$-limit point of a set $A$ then $x$ is $sg$-limit point of $A$.

3.4. Lemma 3.02

(i) If $X$ is $sg-T_0$-space then every $sg-T_0$-limit point and every $sg$-limit point are equivalent.

(ii) If $X$ is $r-T_0$-space then every $sg-T_0$-limit point and every $sg$-limit point are equivalent.

3.5. Theorem 3.03
For $x \neq y \in X$,

(i) $x$ is a $sg-T_0$-limit point of $\{y\}$ iff $x \notin sgcl(y)$ and $y \notin sgcl(x)$.

(ii) $x$ is not a $sg-T_0$-limit point of $\{y\}$ iff either $x \in sgcl(y)$ or $sgcl(x) = sgcl(y)$.

(iii) $x$ is not a $sg-T_0$-limit point of $\{y\}$ iff either $x \notin sgcl(y)$ or $y \notin sgcl(x)$.

3.6. Corollary 3.04

(i) If $x$ is a $sg-T_0$-limit point of $\{y\}$, then $y$ cannot be a $sg$-limit point of $\{x\}$.

(ii) If $sgcl(x) = sgcl(y)$, then neither $x$ is a $sg-T_0$-limit point of $\{y\}$ nor $y$ is a $sg-T_0$-limit point of $\{x\}$.

(iii) If a singleton set $A$ has no $sg-T_0$-limit point in $X$, then $sgclA = sgcl\{x\}$ for all $x \in sgcl(A)$.

3.7. Lemma 3.05

In $X$, if $x$ is a $sg$-limit point of a set $A$, then in each of the following cases $x$ becomes $sg-T_0$-limit point of $A$ ($\langle x \rangle \neq A$).

(i) $sgcl(x) = sgcl(y)$ for $y \in A$, $x \neq y$.

(ii) $sgcl(x) = \{x\}$.

(iii) $x$ is a $sg-T_0$-space.

(iv) $A-\{x\}$ is $sg$-open

4. $sg-T_0$ AND $sg-R_0$ AXIOMS, $i = 0, 1$

In view of Lemma 3.6(iii), $sg-T_0$-axiom implies the equivalence of the concept of limit point of a set with that of $sg-T_0$-limit point of the set. But for the converse, if $x \in sgcl(y)$ then $sgcl(x) = sgcl(y)$ in general, but if $x$ is a $sg-T_0$-limit point of $\{y\}$, then $sgcl(x) = sgcl(y)$.

4.1. Lemma 4.01

In a space $X$, a limit point $x$ of $\{y\}$ is a $sg-T_0$-limit point of $\{y\}$ iff $sgcl(x) \neq sgcl(y)$.

This lemma leads to characterize the equivalence of $sg-T_0$-limit point and $sg$-limit point of a set as $sg-T_0$-axiom.

4.2. Theorem 4.02

The following conditions are equivalent:

(i) $X$ is a $sg-T_0$-space

(ii) Every $sg$-limit point of a set $A$ is a $sg-T_0$-limit point of $A$

(iii) Every $r$-limit point of a singleton set $\{x\}$ is a $sg-T_0$-limit point of $\{x\}$

(iv) For any $x, y \in X$, $x \neq y$ if $x \in sgcl(y)$, then $x$ is a $sg-T_0$-limit point of $\{y\}$

Note 6: In a $sg-T_0$-space $X$ if every point of $X$ is a $r$-limit point of $X$, then every point of $X$ is $sg-T_0$-limit point of $X$. But a space $X$ in which each point is a $sg-T_0$-limit point of $X$ is not necessarily a $sg-T_0$-space.

4.3. Theorem 4.03

The following conditions are equivalent:

(i) $X$ is a $sg-R_0$-space

(ii) For any $x, y \in X$, if $x \in sgcl(y)$, then $x$ is not a $sg-T_0$-limit point of $\{y\}$

(iii) A point $sg$-closure set has no $sg-T_0$-limit point in $X$
4.4. Theorem 4.04  
In a sg-$R_0$ space $X$, a point $x$ is sg-$T_0$-limit point of $A$ if and only if every sg-open set containing $x$ contains infinitely many points of $A$ with each of which $x$ is topologically distinct.

4.5. Theorem 4.05  
$X$ is sg-$R_0$ space if and only if a set $A = \bigcup_{i=1}^{n} \text{sgcl}(x_i)$ a finite union of point closure sets has no sg-$T_0$-limit point. If $sg-R_0$ space is replaced by $R_0$ space in the above theorem, we have the following corollaries:

4.6. Corollary 4.06  
The following conditions are equivalent:
(i) $X$ is a $R_0$ space
(ii) For any $x, y \in X$, if $x \in \text{sgcl}(y)$, then $x$ is not a sg-$T_0$-limit point of $\{y\}$
(iii) A point sg-closure set has no sg-$T_0$-limit point in $X$
(iv) A singleton set has no sg-$T_0$-limit point in $X$.

4.7. Corollary 4.07  
In an $R_0$-space $X$:
(i) If a point $x$ is sg-$T_0$-limit point of a set then every sg-open set containing $x$ contains infinitely many points of $A$ with each of which $x$ is topologically distinct.
(ii) If a point $x$ is sg-$T_0$-limit point of a set then every sg-open set containing $x$ contains infinitely many points of $A$ with each of which $x$ is topologically distinct.
(iii) If $A = \bigcup_{i=1}^{n} \text{sgcl}(x_i)$, a finite union of point closure sets has no sg-$T_0$-limit point.
(iv) If $x \in \text{sgcl}(x_i)$ then $X$ has no sg-$T_0$-limit point. Various characteristic properties of sg-$T_0$-limit points studied so far is enlisted in the following theorem.

4.8. Theorem 4.08  
In a sg-$R_0$-space, we have the following:
(i) A singleton set has no sg-$T_0$-limit point in $X$.
(ii) A finite set has no sg-$T_0$-limit point in $X$.
(iii) A point sg-closure set has no sg-$T_0$-limit point in $X$.
(iv) A union of finite sets has no sg-$T_0$-limit point in $X$.
(v) For any $x, y \in X$, if $x \in \text{sgcl}(y)$, then $x$ is not a sg-$T_0$-limit point of $\{y\}$.
(vi) For any $x, y \in X$, if $x \in \text{sgcl}(y)$, then $x$ is not a sg-$T_0$-limit point of $\{y\}$.
(vii) For any $x, y \in X$, if $x \in \text{sgcl}(y)$, then $x$ is not a sg-$T_0$-limit point of $\{y\}$.
(viii) Any point $x \in X$ is a sg-$T_0$-limit point of a set $A$ in $X$ if and only if every sg-open set containing $x$ contains infinitely many points of $A$ with each of which $x$ is topologically distinct.

4.9. Theorem 4.09  
$X$ is sg-$R_0$ if and only if for any sg-open set $U$ in $X$ and points $x, y$ such that $x \in U, y \in U$, there exists a sg-open set $V$ in $X$ such that $y \in V \subseteq U, x \notin V$.

4.10. Lemma 4.10  
In sg-$R_0$ space $X$, if $x$ is a sg-$T_0$-limit point of $X$, then for any non empty sg-open set $U$, there exists a non empty sg-open set $V$ such that $V \subseteq U, x \notin \text{sgcl}(V)$.

4.11. Lemma 4.11  
In a sg-$T_0$-regular space $X$, if $x$ is a sg-$T_0$-limit point of $X$, then for any non empty sg-open set $U$, there exists a non empty sg-open set $V$ such that $\text{sgcl}(V) \subseteq U, x \notin \text{sgcl}(V)$.

In a regular space $X$:
(i) If $x$ is a sg-$T_0$-limit point of $X$, then for any non empty sg-open set $U$, there exists a non empty sg-open set $V$ such that $\text{sgcl}(V) \subseteq U, x \notin \text{sgcl}(V)$.
(ii) If $x$ is a $T_0$-limit point of $X$, then for any non empty sg-open set $U$, there exists a non empty sg-open set $V$ such that $\text{sgcl}(V) \subseteq U, x \notin \text{sgcl}(V)$.

4.13. Theorem 4.13  
If $X$ is a sg-compact sg-$R_0$-space, then $X$ is a Baire Space.  
Proof: Let $(A_n)$ be a countable collection of sg-closed sets of $X$, each $A_n$ having empty interior in $X$. Take $A_1$, since $A_1$ has empty interior, $A_1$ does not contain any sg-open set say $U_1$. Therefore we can choose a point $y \in U_1$ such that $y \notin A_1$. For $X$ is sg-regular, and $y \notin (X-A_1) \cap U_2$, a sg-open set, we can find a sg-open set $U_2$ in $X$ such that $y \in U_2$, $\text{sgcl}(U_2) \cap (X-A_1) = U_2$. Hence $U_2$ is a non empty sg-open set in $X$ such that $\text{sgcl}(U_2) \subseteq U_1$ and $\text{sgcl}(U_2) \cap A_1 = \emptyset$. Continuing this process, in general, for given non empty sg-open set $U_{n-1}$, we can choose a point of $U_{n-1}$ which is not in the sg-closed set $A_n$ and a sg-open set $U_n$ containing this point such that $\text{sgcl}(U_n) \subseteq U_{n-1}$, and $\text{sgcl}(U_n) \cap A_n = \emptyset$. Thus we get a sequence of nested non empty sg-closed sets which satisfies the finite intersection property. Therefore $\bigcap_{n=1}^{\infty} \text{sgcl}(U_n) \neq \emptyset$. Then some $x \in \bigcap_{n=1}^{\infty} \text{sgcl}(U_n)$ which in turn implies that $x \in U_{n+1}$ as $\text{sgcl}(U_n) \subseteq U_{n+1}$ and $x \notin A_n$ for each $n$.

If $X$ is a compact sg-$R_0$-space, then $X$ is a Baire Space.

4.15. Corollary 4.15  
Let $X$ be a sg-compact sg-$R_0$-space. If $(A_n)$ is a countable collection of sg-closed sets in $X$, each $A_n$ having non-empty sg-interior in $X$, then there is a point of $X$ which is not in any of the $A_n$.

4.16. Corollary 4.16  
Let $X$ be a sg-compact $R_1$-space. If $(A_n)$ is a countable collection of sg-closed sets in $X$, each $A_n$ having non-empty sg- interior in $X$, then there is a point of $X$ which is not in any of the $A_n$.  

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4.17. Theorem 4.17
Let X be a non empty compact sg-R_R-space. If every point of X is a sg-T_T-limit point of X then X is uncountable.

Proof: Since X is non empty and every point is a sg-T_T-limit point of X, X must be infinite. If X is countable, we construct a sequence of sg-open sets \{V_i\} in X as follows;

Let X = V_1, then for x_i \in sg-T_T-limit point of X, we can choose a non empty sg-open set V_2 in X such that V_2 \subset V_1, and x_i \notin sgcl(V_2). Next for x_2 and non empty sg-open set V_2, we can choose a non empty sg-open set V_3 in X such that V_3 \subset V_2 and x_2 \notin sgcl(V_3). Continuing this process for each x_n and a non empty sg-open set V_n, we can choose a non empty sg-open set V_{n+1} in X such that V_{n+1} \subset V_n and x_n \notin sgcl(V_{n+1}).

Now consider the nested sequence of sg-closed sets sgcl(V_1) \supset sgcl(V_2) \supset sgcl(V_3) \supset \ldots \supset sgcl(V_n) \supset \ldots Since X is sg-compact and \{sgcl(V_n)\} the sequence of sg-closed sets satisfies finite intersection property. By Cantors intersection theorem, there exists an x in X such that x \notin sgcl(V_n). Further x \不属于 X, which is not equal to any of the points of X. Hence X is uncountable.

4.18. Corollary 4.18
Let X be a non empty sg-compact sg-R_R-space. If every point of X is a sg-T_T-limit point of X then X is uncountable.

5. \textit{sg-T_T-limit} IDENTIFICATION SPACES AND \textit{sg-separated} AXIOMS

5.1. Definition 5.01
Let (X, \tau) be a topological space and let \% be the equivalence relation on X defined by x\%y iff sgcl(x) = sgcl(y)

5.2. Problem 5.02
Show that x\%y iff sgcl(x) = sgcl(y) is an equivalence relation.

5.3. Definition 5.03
The space (X_0, Q(X_0)) is called the sg-T_T-limit identification space of (X, \tau), where X_0 is the set of equivalence classes of \% and Q(X_0) is the decomposition topology on X_0.

Let P_X: (X, \tau) \rightarrow (X_0, Q(X_0)) denote the natural map.

5.4. Lemma 5.04
If x \in X and A \subseteq X, then x \in sgcl(A) iff every sg-open set containing x intersects A.

5.5. Theorem 5.05
The natural map P_X: (X, \tau) \rightarrow (X_0, Q(X_0)) is closed, and P_X^{-1}(O) = O for all \% \subseteq \tau and (X_0, Q(X_0)) is sg-T_T.

Proof: Let O \subseteq P_X(X, \tau) and let C \subseteq P_X(O). Then there exists x \in O such that P_X(x) \subseteq C, if y \in C, then sgcl(y) = sgcl(x), which, by lemma, implies y \in O. Since \tau \subseteq P_X(X, \tau), then P_X^{-1}(O) = U for all U \subseteq \tau, which implies P_X is closed and open.

Let G, H \subseteq X such that G \neq H, let x \in G and y \in H. Then sgcl(x) \neq sgcl(y), which implies x \notin sgcl(y) or y \notin sgcl(x). Since P_X is continuous and open, then G \neq H \Rightarrow A = P_X(X \setminus sgcl(y)) \notin P_X(X, \tau) and \emptyset \subseteq A.

5.6. Theorem 5.06
The following are equivalent:
(i) X is sg-R_R (ii) X_0 = \{sgcl(x): x \in X\} and (iii) (X_0, Q(X_0)) is sg-T_T.

Proof: (i) \Rightarrow (ii) Let x \in X such that y \in sgcl(x), which implies C \subseteq sgcl(y). If y \in sgcl(x), then x \in sgcl(y), since, otherwise, x \notin sgcl(x) \cap P_X(X \setminus sgcl(y)), which is a contradiction. Thus, if y \notin sgcl(x), then x \notin sgcl(y), which implies sgcl(y) \neq sgcl(x) and y \in C. Hence X_0 = \{sgcl(x): x \in X\}.

(ii) \Rightarrow (iii) Let A = B \subseteq X such that A \subseteq sgcl(y). Then y \in sgcl(y), and A \subseteq P_X(X \setminus sgcl(y)) \subseteq P_X(X, \tau) and B \subseteq C. Then (X_0, Q(X_0)) is sg-T_T.

(iii) \Rightarrow (i) Let x \in U \subseteq X. Then x \notin sgcl(Y), if y \in U \subseteq X, C \subseteq X containing x and y respectively. Then x \notin sgcl(y), which implies C \neq C_y and there exists sg-open set A such that C \subseteq A \subseteq C_y. Since P_X is continuous and open, then y \in B = P_X^{-1}(A) \subseteq X \setminus sgcl(X) and x \notin B, which implies sgcl(y) \subseteq U. This is true for all sgcl(x) \subseteq sgcl(x) \subseteq U. Hence X is sg-R_R.

5.7. Theorem 5.07
(X, \tau) is sg-R_R iff (X_0, Q(X_0)) is sg-T_T.

The proof is straightforward using theorems 5.05 and 5.06 and is omitted.

5.8. Theorem 5.08
X is sg-T_T \iff \exists 0, \exists 1, \exists 2. iff there exists a sg-continuous, almost-open, 1-1 function from (X, \tau) into a sg-T_T space ; \iota = 0, 1, 2, respectively.

5.9. Theorem 5.09
If f: (X, \tau) \rightarrow (Y, \alpha) is sg-continuous, sg-open, and x, y \in X such that sgcl(x) = sgcl(y), then sgcl(f(x)) = sgcl(f(y)).

5.10. Theorem 5.10
The following are equivalent:
(i) (X, \tau) is sg-T_T
(ii) Elements of X_0 are singleton sets
(iii) There exists a sg-continuous, sg-open, 1-1 function f: (X, \tau) \rightarrow (Y, \alpha), where (Y, \alpha) is sg-T_T.

Proof: (i) \Rightarrow (ii) and (iii) \Rightarrow (i) are straightforward and are omitted.

5.11. Corollary 5.11
A space (X, \tau) is sg-T_T \iff \iota = 1, 2 iff (X, \tau) is sg-T_T \iff \iota = 1, 2, respectively, and there exists a sg-continuous, sg-open, 1-1 function f: (X, \tau) \rightarrow \text{into a sg-T_T space}.

5.12. Definition 5.04
\textit{f is point} \rightarrow \textit{sg-closure 1-1 iff for x, y \in X such that sgcl(x) = sgcl(y), sgcl(f(x)) = sgcl(f(y)).

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5.13. Theorem 5.12
(i) If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is point–sg-closure 1–1 and \((X, \tau) \) is sg-T\(_0\), then \( f \) is 1–1
(ii) If \( f : (X, \tau) \rightarrow (Y, \sigma) \), where \((X, \tau)\) and \((Y, \sigma)\) are sg-T\(_1\), then \( f \) is point–sg-closure 1–1 iff \( f \) is 1–1
The following result can be obtained by combining results for sg-T\(_n\)-identification spaces, sg-induced functions and sg-T\(_i\) spaces; \( i = 1, 2 \).

5.14. Theorem 5.13
\( X \) is sg-R\(_i\) ; \( i = 0, 1 \) iff there exists a sg-continuous , almost–open point–sg-closure 1–1 function \( f : (X, \tau) \rightarrow B \) into a sg-R\(_i\) space; \( i = 0, 1 \) respectively.

6. sg-Normal; Almost sg-normal and Mildly sg-normal spaces

6.1. Definition 6.1
A space \( X \) is said to be sg-normal if for any pair of disjoint closed sets \( F_i \) and \( F_j \), there exist disjoint sg-open sets \( U \) and \( V \) such that \( F_i \subseteq U \) and \( F_j \subseteq V \).

Example 4: Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, \{b\}, \{c\}, X\} \). Then \( X \) is sg-normal.

Example 5: Let \( X = \{a, b, c, d\} \) with \( \tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \} \). \( X \) is sg-normal, normal and almost normal. We have the following characterization of sg-normality.

6.2. Theorem 6.1
For a space \( X \) the following are equivalent:
(i) \( X \) is sg-normal.
(ii) For every pair of open sets \( U \) and \( V \) whose union is \( X \), there exist sg-closed sets \( A \) and \( B \) such that \( A \cup U, B \cup V \) and \( A \cup B = X \).
(iii) For every closed set \( F \) and every open set \( G \) containing \( F \), there exists a sg-open set \( U \) such that \( F \subseteq U \subseteq G \).

Proof: (i) \( \Rightarrow \) (ii): Let \( U \) and \( V \) be a pair of open sets in a sg-normal space \( X \). Then \( X = U \cup V \). Since \( X \) is sg-normal, there exist disjoint sg-open sets \( U_j \) and \( V_j \) such that \( X = U_1 \cup U_2 \cup \ldots \cup U_n \cup V_1 \cup V_2 \cup \ldots \cup V_m = X \). Let \( A = X - U_1 \), \( B = X - V_1 \). Then \( A \) and \( B \) are disjoint closed sets such that \( A \cup U, B \cup V \) and \( A \cup B = X \).

6.3. Theorem 6.2
A regular open subspace of a sg-normal space is sg-normal.

Example 6: Let \( X = \{a, b, c, d\} \) with \( \tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \} \). \( X \) is sg-normal and sg-regular. However we observe that every sg-normal sg-R\(_i\) space is sg-regular.

6.4. Definition 6.2
A function \( f : X \rightarrow Y \) is said to be almost–sg-irresolute if for each \( x \in X \) and each sg-neighborhood \( V \) of \( f(x) \), \( sgcl(f^\sim(V)) \) is a sg-neighborhood of \( x \). Clearly every sg-irresolute map is almost sg-irresolute.

The proof of the following lemma is straightforward and hence omitted.

6.5. Lemma 6.1
\( f \) is almost sg-irresolute iff \( f^\sim(V) \subseteq sg-int(sgcl(f^\sim(V))) \) for every \( V \subseteq SGO(Y) \).

6.6. Lemma 6.2
\( f \) is almost sg-irresolute iff \( \bar{f}(V) \subseteq sg-int(sgcl(f^\sim(V))) \) for every \( V \subseteq SGO(X) \).

Proof: Let \( U \subseteq SGO(X) \). Suppose \( y \subseteq sgcl(U) \). Then there exists \( V \subseteq sG O(y) \) such that \( V \subseteq f(V) \subseteq f(U) \). Since \( U \subseteq SGO(X) \), we have \( sg-int(sgcl(f^\sim(V))) \subseteq sgcl(V) \subseteq sgcl(f(U)) \).

6.7. Theorem 6.3
If \( f : X \rightarrow Y \) is M-sg-open continuous almost sg-irresolute, \( X \) is sg-normal, then \( Y \) is sg-normal.

Proof: Let \( A \subseteq X \) be a closed subset of \( X \) and \( Y \) be an open set containing \( A \). Then by continuity of \( f \), \( f(A) \) is closed and \( f(B) \) is an open set of \( Y \) such that \( f(A) \subseteq f(B) \). As \( X \) is sg-normal, there exists a sg-open set \( U \) in \( X \) such that \( f(A) \subseteq U \subseteq sgcl(U) \subseteq f(B) \). Then \( f(B) \subseteq f(U) \subseteq sgcl(U) \subseteq f(B) \).

6.8. Lemma 6.3
A mapping \( f \) is M-sg-closed if and only if for each subset \( B \) in \( Y \) and for each sg-open set \( U \) in \( X \) containing \( f(B) \), there exists a sg-open set \( V \) containing \( B \) such that \( f(V) \supseteq U \).

6.9. Theorem 6.4
If \( f : X \rightarrow Y \) is M-sg-closed continuous, \( X \) is sg-normal space, then \( Y \) is sg-normal.

Proof of the theorem is routine and hence omitted.
Now in view of lemma 2.2 [9] and lemma 6.3, we prove that the following result.

6.10. Theorem 6.5
If \( f \) is an M-sg-closed map from a weakly Hausdorff sg-normal space \( X \) onto a space \( Y \) such that \( f(y) \) is S-closed relative to \( X \) for each \( y \in Y \), then \( Y \) is sg-T\(_2\).

Proof: Let \( y_1 \neq y_2 \in Y \). Since \( X \) is weakly Hausdorff, \( f^\sim(y_1) \) and \( f^\sim(y_2) \) are disjoint closed subsets of \( X \) by lemma 2.2 [9]. As \( X \) is sg-normal, there exist disjoint \( \forall V \subseteq SGO(X) \) such that \( f^\sim(y_i) \subseteq V_i \) for \( i = 1, 2 \). Since \( f \) is M-sg-closed, there exist disjoint \( U \subseteq SGO(Y, y_1) \) and \( f^\sim(U) \subseteq V_i \) for \( i = 1, 2 \). Hence \( Y \) is sg-T\(_2\).

6.11. Theorem 6.6
For a space \( X \) we have the following:

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http://www.discovery.org.in/iije.htm
(a) If X is normal then for any disjoint closed sets A and B, there exist disjoint sg-open sets U and V such that A⊂U and B⊂V.
(b) If X is normal then for any closed set A and any open set V containing A, there exists an sg-open set U of X such that A⊂U⊂sgcl(U)⊂V.

6.12. Definition 6.2
X is said to be almost sg-normal if for each closed set A and each regular closed set B such that A∩B = ∅, there exist disjoint sg-open sets U and V such that A⊂U and B⊂V.

Clearly, every sg-normal space is almost sg-normal, but not conversely in general.

6.13. Theorem 6.7
For a space X the following statements are equivalent:
(i) X is almost sg-normal
(ii) For every pair of sets U and V, one of which is open and the other is regular open whose union is X, there exist sg-closed sets G and H such that G⊂U, H⊂V and G∩H = ∅.
(iii) For every closed set A and every regular open set B containing A, there is a sg-open set V such that A⊂V⊂sgcl(V)⊂B.

Proof: (i)⇒(ii) Let U be an open set and V be a regular open set in almost sg-normal space X such that U∪V = X. Then (X−U) is a closed set and (X−V) is regular closed with (X−U)∩(X−V) = ∅. By almost sg-normality of X, there exist disjoint sg-open sets U₁ and V₁ such that X−U ⊂ U₁ and X−V ⊂ V₁. Let G = X−U₁ and H = X−V₁. Then G and H are sg-closed sets such that G⊂U, H⊂V and G∩H = ∅.

(ii)⇒(i) and (iii)⇒(ii) are obvious.

6.14. Theorem 6.8
Every almost regular, sg-compact space X is almost sg-normal.

Recall that a function \( f : X \to Y \) is called rc-continuous if inverse image of regular closed set is regular closed.

Now, we state the invariance of almost sg-normality in the following.

6.15. Theorem 6.9
If \( f \) is continuous M-sg-open rc-continuous and almost sg-irresolute surjection from an almost sg-normal space X onto a space Y, then Y is almost sg-normal.

6.16. Definition 6.3
A space X is said to be mildly sg-normal if for every pair of disjoint regular closed sets \( F₁ \) and \( F₂ \) of X, there exist disjoint sg-open sets U and V such that \( F₁ \subset U \) and \( F₂ \subset V \).

Example 7: Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\} \). Then X is mildly sg-normal.

We have the following characterization of mild sg-normality.

6.17. Theorem 6.10
For a space X the following are equivalent:
(i) X is mildly sg-normal.
(ii) For every pair of regular open sets U and V whose union is X, there exist sg-closed sets G and H such that G⊂U, H⊂V and G∩H = ∅.
(iii) For any regular closed set A and every regular open set B containing A, there exists a sg-open set U such that A⊂U⊂sgcl(U)⊂B.
(iv) For every pair of disjoint regular closed sets, there exist sg-open sets U and V such that A⊂U, B⊂V and sgcl(U)∩sgcl(V) = ∅.

This theorem may be proved by using the arguments similar to those of Theorem 6.7.

Also, we observe that mild sg-normality is regular open hereditary.

6.18. Definition 6.4
A space X is weakly sg-regular if for each point \( x \) and a regular open set \( U \) containing \( x \), there is a sg-open set \( V \) such that \( x∈V⊂clV⊂U \).

Example 8: Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\} \). Then X is weakly sg-regular.

Example 9: Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, \{a\}, \{a, b\}, \{a, b\}, \{a, b, c\}, X\} \). Then X is not weakly sg-regular.

6.19. Theorem 6.11
If \( f : X \to Y \) is an M-sg-open rc-continuous and almost sg-irresolute function from a mildly sg-normal space X onto a space Y, then Y is mildly sg-normal.

Proof: Let A be a regular closed set and B be a regular open set containing A. Then by rc-continuity of \( f \), \( f^{-1}(A) \) is a regular closed set contained in the regular open set f(V). Since X is mildly sg-normal, there exists a sg-open set V such that \( f^{-1}(A)⊂clV⊂sgcl(V)⊂f^{-1}(B) \). By Theorem 6.10. As \( f \) is M-sg-open and almost sg-irresolute surjection, \( f(V)⊂SGO(Y) \) and \( A⊂f(V)⊂sgcl(f(V))⊂B \). Hence Y is mildly sg-normal.

6.20. Theorem 6.12
If \( f : X \to Y \) is rc-continuous, M-sg-closed map and X is mildly sg-normal space, then Y is mildly sg-normal.

7. sg-US SPACES

7.1. Definition 7.1
A sequence \( <xₙ> \) is said to be sg-converges to a point \( x \) of X, written as \( <xₙ> \to_S x \) if \( <xₙ> \) is eventually in every sg-open set containing \( x \).

Clearly, if a sequence \( <xₙ> \to_S x \) converges to a point \( x \) of X, then \( <xₙ> \to_{SGO} x \) converges to \( x \).

7.2. Definition 7.2
X is said to be sg-US if every sequence \( <xₙ> \) in X sg-converges to a unique point.

7.3. Definition 7.3
A set \( F \) is sequentially sg-closed if every sequence in \( F \) sg-converges to a point in \( F \).

7.4. Definition 7.4
A subset \( G \) of a space \( X \) is said to be sequentially sg-compact if every sequence in \( G \) has a subsequence which sg-converges to a point in \( G \).

7.5. Definition 7.5
A point y is a sg-cluster point of sequence \( <x_n> \) iff \( <x_n> \) is frequently in every sg-open set containing x. The set of all sg-cluster points of \( <x_n> \) will be denoted by sg-cl(\( x \nbar \)).

7.6. Definition 7.6
A point y is sg-side point of a sequence \( <x_n> \) if y is a sg-cluster point of \( <x_n> \) but no subsequence of \( <x_n> \) converges to y.

7.7. Definition 7.7
A space X is said to be
(i) sg-S; if it is sg-US and every sequence \( <x_n> \) sg-converges with subsequence of \( <x_n> \) sg-side points.
(ii) sg-S; if it is sg-US and every sequence \( <x_n> \) in X sg-converges which has no sg-side point.

Using sequentially continuous functions, we define sequentially sg-continuous functions.

7.8. Definition 7.8
A function f is said to be sequentially sg-continuous at \( x \in X \) if \( f(x_n) \rightarrow y \) whenever \( x_n \rightarrow x \). If f is sequentially sg-continuous at all \( x \in X \), then f is said to be sequentially sg-continuous.

7.9. Theorem 7.1
We have the following:
(i) Every sg-T2 space is sg-US.
(ii) Every sg-US space is sg-T1.
(iii) X is sg-US iff the diagonal set is a sequentially sg-closed subset of \( X \times X \).
(iv) X is sg-T2 iff it is both sg-R1 and sg-US.
(v) Every regular open subset of a sg-US space is sg-US.
(vi) Product of arbitrary family of sg-US spaces is sg-US.
(vii) Every sg-S2 space is sg-S1 and Every sg-S1 space is sg-US.

7.10. Theorem 7.2
In a sg-US space every sequentially sg-compact set is sequentially closed.

Proof: Let X be sg-US space. Let Y be a sequentially sg-compact subset of X. Let \( <x_n> \) be a sequence in Y. Suppose that \( <x_n> \) sg-converges to a point in X-Y. Let \( <x_{n_j}> \) be subsequence of \( <x_n> \) such that \( x_{n_j} \rightarrow y \) in X-Y since Y is sequentially sg-compact. Also, let a subsequence \( <x_{n_{j_k}}>, \) of \( <x_{n_j}> \) be a sequence in Y. Since \( x_{n_{j_k}} \in X \) is in the sg-US space X, \( x \rightarrow y \). Thus, Y is sequentially sg-closed.

8. SEQUENTIALLY SUB-sg-CONTINUITY
In this section we introduce and study the concepts of sequentially sub-sg-continuity, sequentially nearly sg-continuity and sequentially sg-compact preserving functions and study their relations and the property of sg-US spaces.

8.1. Definition 8.1
A function f is said to be
(i) sequentially nearly sg-continuous if for each point \( x \in X \) and each sequence \( <x_n> \rightarrow x \) in X, there exists a subsequence \( <x_{n_k}> \) of \( <x_n> \) such that \( f(x_{n_k}) \rightarrow f(x) \).
(ii) sequentially sub-sg-continuous if for each point \( x \in X \) and each sequence \( <x_n> \rightarrow x \) in X, there exists a subsequence \( <x_{n_k}> \) of \( <x_n> \) and a point \( y \in Y \) such that \( <f(x_{n_k})> \rightarrow f(y) \).
(iii) sequentially sg-compact preserving if \( f(K) \) is sequentially sg-compact in Y for every sequentially sg-compact set K of X.

8.2. Lemma 8.1
Every function f is sequentially sg-compact if Y is a sequentially sg-compact.

Proof: Let \( <x_n> \rightarrow x \) in X. Since Y is sequentially sg-compact, there exists a subsequence \( \{f(x_{n_k})\} \) of \( \{f(x_n)\} \) converging to a point \( y \in Y \). Hence f is sequentially sub-sg-compact.

8.3. Theorem 8.1
Every sequentially sg-continuous function is sequentially sg-compact.

Proof: Assume f is sequentially sg-continuous and K is any sequentially sg-compact subset of X. Let \( <y_j> \) be any sequence in f(K). Then for each positive integer \( n_1 \), there exists a point \( x_{n_1} \in K \) such that \( f(x_{n_1}) = y_j \). Since \( <x_{n_1}> \) is a sequence in the sequentially sg-compact set K, there exists a subsequence \( <x_{n_{1_k}}> \) of \( <x_{n_1}> \) sg-converging to a point \( x \in K \). By hypothesis, f is sequentially sg-continuous and hence there exists a subsequence \( <y_{j_k}> \) of \( <y_j> \) such that \( f(x_{n_{1_k}}) \rightarrow f(x) \). Thus, there exists a subsequence \( <y_{j_k}> \) of \( <y_j> \) sg-converging to f(x) \in f(K). This shows that f(K) is sequentially sg-compact set in Y.

8.4. Theorem 8.2
Every sequentially s-continuous function is sequentially sg-continuous.

Proof: Let f be a sequentially s-continuous and \( <x_n> \rightarrow x \) in X. Then \( <x_n> \rightarrow x \). Since f is sequentially s-continuous, \( f(x_n) \rightarrow f(x) \). But we know that \( <x_n> \rightarrow x \) implies \( <x_n> \rightarrow x \) and hence \( f(x_n) \rightarrow f(x) \) implies f is sequentially sg-continuous.

8.5. Theorem 8.3
Every sequentially sg-compact preserving function is sequentially sub-sg-continuous.

Proof: Suppose f is a sequentially sg-compact preserving function. Let x be any point of X and \( <x_n> \) any sequence in X sg-converging to x. We shall denote the set \( \{x_n \mid n \geq 1,2,3, \ldots \} \) by A. and K = A \cup \{x\}. Then K is sequentially sg-compact since \( (x_n) \rightarrow x \). By hypothesis, f is sequentially sg-compact preserving and hence f(K) is a sequentially sg-compact set of Y. Since \( f(x_n) \) is a sequence in f(K), there exists a subsequence \( \{f(x_{n_k})\} \) of \( \{f(x_n)\} \) sg-converging to a point \( y \in f(K) \). This implies that f is sequentially sub-sg-continuous.
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A function \( f: X \to Y \) is sequentially sg-compact preserving iff \( f^{-1}(K) \) is sequentially sub-sg-continuous for each sequentially sg-compact subset \( K \) of \( X \).

**Proof:** Suppose \( f \) is a sequentially sg-compact preserving function. Then \( f(K) \) is sequentially sg-compact set in \( Y \) for each sequentially sg-compact set \( K \) of \( X \). Therefore, by Lemma 8.1 above, \( f^{-1}(K) \) is sequentially sg-continuous function.

Conversely, let \( K \) be any sequentially sg-compact set of \( X \). Let \( \{x_n\} \) be any sequence in \( f(K) \). Then for each positive integer \( n \), there exists a point \( x_n \in K \) such that \( f(x_n) = y_n \). Since \( \{x_n\} \) is a sequence in the sequentially sg-compact set \( K \), there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) sg-converging to a point \( x \in K \). By hypothesis, \( f^{-1}(K) \) is sequentially sub-sg-continuous and hence there exists a subsequence \( \{y_{n_k}\} \) of \( \{y_n\} \) sg-converging to a point \( y \in f(K) \). This implies that \( f(K) \) is sequentially sg-compact set in \( Y \). Thus, \( f \) is sequentially sg-compact preserving function.

The following corollary gives a sufficient condition for a sequentially sg-continuous function to be sequentially sg-compact preserving.

**8.7. Corollary 8.1**

If \( f \) is sequentially sub-sg-continuous and \( f(K) \) is sequentially sg-closed set in \( Y \) for each sequentially sg-compact set \( K \) of \( X \), then \( f \) is sequentially sg-compact preserving function.

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