RESEARCH

4(10), April 1, 2013

ISSN 2278–5469 EISSN 2278–5450

Discovery

Observations on the hyberbola ax[2] - (a+1) y2 =3a-1

Vidhyalakshmi S[1], Gopalan MA[2☼], kavitha A[3]

1.Professor, Department of Mathematics SIGC, Trichy-2, Tamil Nadu, India
2.Professor, Department of Mathematics SIGC, Trichy-2, Tamil Nadu, India
3.Lecturer, Department of Mathematics SIGC, Trichy-2, Tamil Nadu, India

☼Corresponding Author: Gopalan MA, Professor, Department of Mathematics, Shri mati Indira Gandhi College, Trichy-2. E-mail: mayilgopalan@gmail.com

Publication History
Received: 23 February 2013
Accepted: 27 March 2013
Published: 1 April 2013

Citation
Vidhyalakshmi S, Gopalan MA, kavitha A. Observations on the hyberbola ax[2] - (a+1) y2 =3a-1. *Discovery*, 2013, 4(10), 22-24

ABSTRACT

Knowing an integral point on the hyperbola $ax^2-(a+1)y^2=3a-1$ a process of generating sequence of integral points based on the known solution of the hyperbola is illustrated. A few interesting properties among the solutions are presented.

Key Words: Binary quadratic equation, Integral solutions

MSC 2000 subject classification no: 11D09

1. INTRODUCTION

The binary quadratic diophantine equation of the form $y^2 = Dx^2 + N$, Where D is a non-square positive integer has been studied by various mathematicians for its non- trivial integral solutions where D takes different numerical values (Dickson, 1952, Mordell, 1969, Telng, 1996, Mollin, 1998). In this context, one may also refer (Gopalan et al. 2001, 2007 a, b, c., 2008 a, b, c, d). However in (Gopalan, 2007d) it is shown that the hyperbola represented by $3x^2 + xy = 14$ has only finite number of integral points. These results have motivated us to search for other choices of hyperbolas having infinitely many non- zero integral solutions. It is towards this end, we search for infinitely many non – zero integral solutions on the hyperbola given by $ax^2-(a+1)y^2=3a-1$. In particular, knowing an integral point on the hyperbola $ax^2-(a+1)y^2=3a-1$ a process of generating sequence of integral points based on the known solution of the hyperbola is illustrated. A few interesting properties among the solutions are presented.

2. METHOD OF ANALYSIS

The binary quadratic equation representing the hyperbola is

$$ax^2-(a+1)y^2=3a-1 \quad (1)$$

Introduction of the linear transformations

$$\left.\begin{aligned} x&=X+(a+1)T\\ y&=X+aT\end{aligned}\right\} \quad (2)$$

in (1) leads to

$$X^2=a(a+1)T^2-3a+1 \quad (3)$$

whose initial solution is $(X_0, T_0)=(a-1,1)$

Employing the integral solutions of the Pell's equation

$$X^2=a(a+1)T^2+1 \quad (4)$$

and applying the lemma of Brahmagupta, the non-zero distinct integral solutions of (1) are found to be

$$x_{n+1}=af+\left(\frac{2a^2+a-1}{2.\sqrt{(a^2+a)}}\right)g$$

$$y_{n+1}=\left(\frac{2a-1}{2}\right)f+\left(\frac{a^2}{\sqrt{a^2+a}}\right)g$$

where n = -1,0,1.........

$$f=\left[(2a+1)+2\sqrt{a^2+a}\right]^{n+1}+\left[(2a+1)-2\sqrt{a^2+a}\right]^{n+1}$$

$$g=\left[(2a+1)+2\sqrt{a^2+a}\right]^{n+1}-\left[(2a+1)-2\sqrt{a^2+a}\right]^{n+1}$$

The recurrence relations satisfied by x_{n+1}, y_{n+1} are correspondingly exhibited below:

$$x_{n+3}-(4a+2)x_{n+2}+x_{n+1}=0, \quad x_0=2a, \quad x_1=4a^2+4a-2$$

$$y_{n+3}-(4a+2)y_{n+2}+y_{n+1}=0, \quad y_0=2a-1, \quad y_1=8a^2-1$$

2.1. Generation of solutions

Let (x_0, y_0) be any given integer solution of (1)

Asume that

$$\left.\begin{aligned} x_1&=x_0+h\\ y_1&=h-y_0\end{aligned}\right\} \quad (5)$$

be the second solution of (1). Substituting (5) in (1) and simplifying we get,

$$h=2ax_0+2y_0(a+1) \quad (6)$$

using (6) in (5) the integral solution of (1) are written in the matrix form as

$$\begin{pmatrix} x_1\\ y_1\end{pmatrix}=\begin{pmatrix} 2a+1 & 2(a+1)\\ 2a & (2a+1)\end{pmatrix}\begin{pmatrix} x_0\\ y_0\end{pmatrix}$$

The repetition of the above process leads to the general solution (x_{n+1}, y_{n+1}) of (1), represented by

$$\begin{pmatrix} x_{n+1}\\ y_{n+1}\end{pmatrix}=\begin{pmatrix} X_n & (a+1)Y_n\\ aY_n & X_n\end{pmatrix}\begin{pmatrix} x_0\\ y_0\end{pmatrix}$$

Where (X_n, Y_n) satisfies (4)

A few interesting properties among the solutions are exhibited below:

1) $(a^2+a)[2(2a-1)x_{n+1}-4ay_{n+1}]^2$ is written as the difference of two square

2) $[2a^2x_{n+1}-(2a^2+a-1)y_{n+1}]^2-(a^2+a)[(2a-1)x_{n+1}-2ay_{n+1}]^2$ is a perfect square.

3) $4a^2x_{3n+3}-(4a^2+2a-2)y_{n+3}+12a^2x_{n+1}-6(2a^2+a-1)y_{n+1}\equiv 0 \bmod (3a-1)$

4) $(3a-1)^2\left\{2(3a-1)[2a^2x_{4n+4}-(2a^2+a-1)y_{4n+4}]+16\{[2a^2x_{n+1}-(2a^2+a-1)y_{n+1}]^2\}\right\}$ is a biquadratic integer

5) Each of the following expressions is a Nasty number

i. $6\{(a^2+a)[(4a-2)x_{n+1}-4ay_{n+1}]^2+4(3a-1)^2\}$

ii. $6\{[4a^2x_{n+1}-(4a^2+2a-2)y_{n+1}]^2-4(3a-1)^2\}$

iii. $6(3a-1)[4a^2x_{2n+2}-(4a^2+2a-2)y_{n+2}+2(3a\_1)]$

iv. $6(3a-1)[4a^2x_{2n+2}-(4a^2+2a-2)y_{n+2}-2(3a\_1)]$

v. $6(3a-1)[4a^2x_{4n+4}-2(2a^2+a-1)y_{4n+4}+2(3a\_1)]$

vi. $6\{2[2a^2x_{n+1}-(2a^2+a-1)y_{n+1}]^2+2(a^2+a)^2[(2a-1)x_{n+1}-2ay_{n+1}]^2\pm 2(3a-1)^2\}$

3. CONCLUSION

It is worth to mention here that, instead of (2), one may also consider the linear transformations as $x = X-(a+1)T, y = X-aT$ and obtain a different choice of integral solutions. To conclude, one may search for other forms of hyperbolas to obtain their infinitely many non-zero distinct integral solutions.

REFERENCE

1. Dickson LE. History of Theory of Numbers, Chelsea Publishing Company, New York, Vol II, 1952
2. Mordell LJ. Diophantine equations, Academic Press, New York, 1969
3. Telang SG. Number Theory, Tata McGraw-Hill Publishing Company Limited, New Delhi, 1996
4. Mollin RA. All Solutions of the Diophantine Equation $X^2-Dy^2=n$, *Far East J.Math.Sci.,Special*, 1998, Part III, p.257-293
5. Gopalan MA, Srikanth R. On a Binary quadratic Diophantine Equation $Y^2=DX^2+N$, *Bulletin of Pure and Applied Sciences*, 2001, 20E(1), 7-11
6. Gopalan MA, Anbuselvi R. Integral solutions of $x^2+pxy+y^2=4N$, *Proc.Nat.Acad.Sci.India*, 2007a, 77(A), III, 255-257
7. Gopalan MA, Anbuselvi R. On the Diophantine equation $x^2+bxy+cY^2=1$, *Acta Ciencia Indica*, 2007b, XXXIIIM, 4, 1785-1787
8. Gopalan MA, Gokila K, Vidhyalakshmi S. On The Diophantine Equation $x^2+4xy+y^2+2x+2y+6=0$, *Acta Ciencia Indica*, 2007c, XXXIIIM, 2, 567-570
9. Gopalan MA, Vidhyalakshmi S, Devibala S. On The Diophantine Equation $3x^2+xy=14$, *Acta Ciencia Indica*, 2007d, XXXIIIM, 2, 645-646
10. Gopalan MA, Janaki G. Observation on $Y^2=3X^2+1$, *Acta Ciencia Indica*, 2008a, XXXIVM, 2, 693-695
11. Gopalan MA, Anbuselvi R. Integral solutions of $4ay^2-(a-1)x^2=3a+1$, *Acta Ciencia Indica*, 2008b, XXXIVM, 1, 291-295
12. Gopalan MA, Janaki G. Observation on $x^2-y^2+x+y+xy=2$, *Impact J. Sci. Tech*, 2008c, 2(3), 143-148
13. Gopalan MA, Shanmuganadham P, Vijayashankar A. On Binary Quadratic Equation $x^2-5xy+y^2+8x-20y+15=0$, *Acta Ciencia Indica*, 2008d, XXXIVM, 4, 1803-1805