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A study of functors associated with fuzzy modules

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ABSTRACT

In this paper first we construct a functor then we study this functor as well as we investigate this functor associated with fuzzy modules.

In this paper we show that:-

1.the R-modules and R-homomorphisms form a category, where R be a ring. This category is denoted by 'CM';

2.the fuzzy left R-module and fuzzy R-map form a category. This category is denoted by 'CF';

3.Ω: 'CM' → 'CF' is an covariant functor;

4.is also a homotopy type invariant functor.

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1. INTRODUCTION

The concept of fuzzy modules was introduced by Nogoita and Ralescu [1] and the category of fuzzy sets was introduced by Goguen[2] in 1967. In this paper we establish the function spaces associated with fuzzy modules.

To do this we recall the following definitions and statements.

Definition 1.1

Let R be a ring and M be left or right R-module. (M,λ) is called a **fuzzy left R-module** if there is a map $\lambda : M \to [0,1]$ satisfying the following conditions:

i) $\lambda(a+b) \geq \min\{\lambda(a), \lambda(b)\}, (\forall\ a,b \in M)$
ii) $\lambda(-a) = \lambda(a), \forall\ a \in M$
iii) $\lambda(0) = 1$
iv) $\lambda(ra) = \lambda(a)\ (\forall\ a \in M, r \in R)$

We write (M,λ) by λ_M

Definition 1.2

Let λ_M and η_N be arbitrary fuzzy left R-modules. A **fuzzy R-map**

$\tilde{f} : \lambda_M \to \eta_N$ should satisfy the following conditions.

i) $f : M \to N$ is an R-map,
ii) $\eta(f(a)) \geq \lambda(a),\ \forall\ a \in M$

Definition 1.3

Let $f:M \to N$ and μ be a fuzzy subset of N . The fuzzy subset $f^{-1}(\mu)$ of N defined as follows;
for all $x \in M$, $f^{-1}(\mu)(x) = \mu(f(x))$ is called fuzzy preimage of μ under f.

Definition 1.4

A fuzzy submodules of M is a fuzzy subset of M such that
i)$\mu(0) = 1$
ii)$\mu(rx) \geq \mu(x),\ \forall\ r \in R$ and $\forall x \in M$
iii)$\mu(x+y) \geq \min(\mu(x),\mu(y)),\ \forall x, y \in M$

Definition 1.5

A fuzzy R-map $\tilde{f} \in Hom(\lambda_M,\eta_P)$ is called fuzzy split iff there exists some $\tilde{g} \in Hom(\eta_P,\lambda_M)$ such that $\tilde{f}\tilde{g} = \widetilde{I_p}$ and $\tilde{g}\tilde{f} = \widetilde{I_M}$,

Definition 1.6

A category C consists of
(a) a class of objects X,Y ,Z,....,denoted by Ob(C);
(b) for each ordered pair of objects X,Y a set of morphisms with domain X and range Y denoted by C(X,Y);
(c) for each order triple of objects X,Y and Z and a pair of morphisms $f : X \to Y$ and $g : Y \to Z$ their composite is denoted by $g\,f : X \to Z$, satisfying the following two axioms:
i) associativity : if $f \in C(X,Y)$ and $g \in C(Y,Z)$ and $h \in C(Z,W)$,then $h(gf) = (hg)f \in C(X,W)$
ii) identity : for each object Y in C there is a morphism $I_Y \in C(Y,Y)$ such that if $f \in C(X,Y)$ then $I_Y f = f$ and if $h \in C(Y,Z)$, then $h\, I_Y = h$.

Definition 1.7

Let C and D be categories. A covariant functor T from C to D consists of
i) an object function which assigns to every object X of C an object T(X) of D; and
ii) a morphism function which assigns to every morphism $f : X \to Y$ in C a morphism $T(f) : T(X) \to T(Y)$ in D such that
a)$T(I_X) = I_{T(X)}$
b)$T(gf) = T(g).T(f)$, for $g : Y \to W$ in C.

Definition 1.8

A base point preserving continuous map $f: X \to Y$ is a homotopy equivalence if there is a base point preserving continuous map $g : Y \to X$ with $g \circ f \simeq I_X$ and $f \circ g \simeq I_Y$.

Lemma 1.9

Let $Hom(\lambda_M,\eta_N)$ denotes the set of all fuzzy R-maps from λ_M to η_N , then $Hom(\lambda_M,\eta_N)$ is an additive group. Moreover, if R is a commutative ring, then $Hom(\lambda_M,\eta_N)$ is a left R-modules.

Proof: Using [10],it follows.

Lemma 1.10
For a commutative ring R, R-modules M and N, Hom(M,N) is an R-module

Proof: Using [2], it follows.

Lemma 1.11
Let Hom(M,N) denotes the set of all fuzzy R-maps from R-modules M to R-modules N, then Hom(M,N) is an R –module, if R is a commutative ring

Proof
Using definition 1.1 and [7], it follows

Lemma 1.12
Given a fixed R-module M, the R-homomorphism $f:N \to P$ induces

a) an R-homomorphism $f_*:Hom(M,N) \to Hom(M,P)$ defined by $f_*(\alpha) = f \circ \alpha, \forall \alpha \in Hom(M,N)$
b) an R-homomorphism $f^*:Hom(P,M) \to Hom(N,M)$ defined by $f^*(\beta) = \beta \circ f, \forall \beta \in Hom(P,M)$

Proof:
Using [6,7], it follows

Lemma1.13
Let M,N,P be R-modules and $f:M \to N$ and $g:N \to P$ be R-homomorphisms.
Then for any R- module A

i) $(g \circ f)_*: Hom(A,M) \to Hom(A,P)$ is an R-homomorphism such that
$(g \circ f)_* = g_* \circ f_*$;
ii) $(g \circ f)^*: Hom(P,A) \to Hom(M,A)$ is an R-homomorphism such that
$(g \circ f)^* = f^* \circ g^*$;

Proof:
Using [6,7], it follows

2. CATEGORIES OF FUZZY MODULES
In this section we construct some categories associated with fuzzy modules.

Proposition 2.1
Let R be a ring and M be left or right R-module. Then **R-modules** and **R-homomrphisms** forms a category. This category is denoted by **'CM'**

Proof:
We take all left R-modules of ***'CM'*** as the set of objects and the set of their R- homomorphisms, the set of morphisms Hom(M,N) and for every pair of objects (M,N) and (N,P), the compositions $Hom(M,N) \times Hom(N,P)$, denoted by
$(f , g) = g \circ f$, where $f \in Hom(M,N)$ and $g \in Hom(N,P)$,satisfying the following axioms:

a) for any object $M \in$ obj.'CM', there exists an identity morphism $I_M \in Hom(M,M)$;
b) associativity of the composition holds.

Proposition 2.2
Let R be a ring and M be left or right R-module. Then **fuzzy left R-module** and **fuzzy R-map** forms a category. This category is denoted by **'CF'**

Proof:
We take all elements of 'C***F'*** as the set of objects and the set of their fuzzy R- maps, the set of morphisms $Hom(\lambda_M,\eta_N)$ and for every pair of objects (λ_M,η_N) and (λ_N,η_P) ,the compositions $Hom(\lambda_M,\eta_N) \times Hom(\lambda_N,\eta_P)$,denoted by $(\tilde{f}, \tilde{g}) = \widetilde{g.f} = \tilde{g} \circ \tilde{f}$, where $\tilde{f} \in Hom(\lambda_M,\eta_N)$ and
$\tilde{g} \in Hom(\lambda_N,\eta_P)$, satisfying the following axioms:

a) for any object $\lambda_M \in$ obj.'CM', there exists an identity morphism $\widetilde{I}_M \in Hom(\lambda_M,\lambda_M)$

b) associativity of the composition holds.

Proposition 2.3

Let M and N be arbitrary fuzzy left R-modules. A fuzzy R-map f:M→N is an fuzzy R-homomorphism, then for any fuzzy sub-module S of N ,then

i) $f(O_M) = O_N$ and $f(-x) = -f(x)$, $\forall x \in M$; and

ii) the set $f^{-1}(S) = \{x \in M: f(x) \in S\}$is a fuzzy sub-module of M.

Proof:

i) Now $f(O_M) = f(O_M + O_M) = f(O_M) + f(O_M) \Rightarrow f(O_M) = O_N$.

Again $f(O_M) = f(x + (-x)) = f(x) + f(-x) = O_N \Rightarrow f(-x) = -f(x)$.

ii) Since $f(O_M) = O_N \in S \Rightarrow O_M \in f^{-1}(S) \Rightarrow f^{-1}(S) \neq \Phi$. Let $x,y \in f^{-1}(S) \Rightarrow f(x), f(y) \in S \Rightarrow$ $f(x-y) = f(x) - f(y) \in S$, since S Iis a submodule of N $\Rightarrow x-y \in f^{-1}(S)$. Similarly, for $r \in R$ and $x \in f^{-1}(S)$, $rx \in f^{-1}(S) \Rightarrow f^{-1}(S)$ is a submodule of M. Since M is a fuzzy modules and hence by **definition 1.1 and 1.4**, it follows that $f^{-1}(S)$ is a submodule of M.

Proposition 2.4

Let **CF** denotes the category of fuzzy R-modules and fuzzy R-maps and **CM** denotes the category of R-modules and R-homomorphisms, then there exists a covariant functor

λ: **CM** → C**F**

Proof:

Define λ: **CM** → **CF** by

$\lambda(M) = (M,\lambda) = \lambda_M$, which is the object of **CF**

Let M.N are two R-modules in **CM** and f : M→N be R-homomorphisms in **C**M, then

$\lambda(f) : \lambda(M) \to \lambda(N)$ in CF.

$\lambda(f)(\alpha) = \alpha \cdot f^{-1}$, $\forall \alpha$ in $\lambda(M)$

i) $\alpha(a+b) \geq \min\{\alpha(a), \alpha(b)\}, (\forall a,b \in M)$

ii) $\alpha(-a) = \alpha(a)$,$\forall a \in M$

iii) $\alpha(0) = 1$

iv) $\alpha(ra) = \alpha(a)$ $(\forall a \in M, r \in R)$

Let μ in N, then $f^{-1}(\mu)(x) = \mu(f(x)) \Rightarrow \alpha_1(f^{-1}(\mu))(x) = \alpha_1(\mu f(x)) = (\alpha_1\mu)(f(x))$

$\Rightarrow \alpha_2(f^{-1}(\mu))(x) = \alpha_2(\mu f(x)) = \alpha_2\mu(f(x))$

Thus $\alpha_1 = \alpha_2 \Rightarrow \alpha_1 \cdot f^{-1} = \alpha_2 \cdot f^{-1} \Rightarrow \lambda(f)(\alpha_1) = \lambda(f)(\alpha_2)$

Let f:M→N and g: N→P are in CM, then $\lambda(f) : \lambda(M) \to \lambda(N)$,

$\lambda(g) : \lambda(N) \to \lambda(P)$ and gof : M → P are in **C**M.

Now $\lambda(g \circ f) : \lambda(M) \to \lambda(P)$ by $\lambda(g.f)(\alpha) = \alpha(gf)^{-1} = \alpha(f^{-1}. g^{-1}) = (\alpha f^{-1}) g^{-1} = (\lambda(f)(\alpha)) g^{-1} = \lambda(f)(\alpha g^{-1}) = (\lambda(f) \lambda(g))(\alpha)$ are in **CF** , also $\lambda(g) \lambda(f) : \lambda(M) \to \lambda(P)$ in **CF** $\Rightarrow$

$\lambda(g \circ f) = \lambda(g) \lambda(f)$.

Also $\lambda(I_M) = I_{\lambda(M)} \Rightarrow \lambda$: **CM** → **CF** is a covariant functor

Proposition 2.5

Let R be a ring and M be a fixed R- module, then $Hom_R(M,N)$ is a fuzzy R-module, for any R-module N.

Proof

Using Definition 1.1 and [7], it follows.

Proposition 2.6

Let R be a ring and M be a fixed R- module, **the** R-homomorphism f:N→P induces

i) an fuzzy R-homomorphism f_* : $Hom_R(M,N) \to Hom_R(M,P)$ and

ii) an fuzzy R-homomorphism f^* : $Hom_R(P,M) \to Hom_R(N,M)$

Proof: Using Definition 1.1 and Lemma 1.12, it follows.

Corollary 2.7

For any fixed R-module M, the fuzzy R module $Hom_R(M,N)$ and their fuzzy R-homomorphisms forms category, for any R-module N; this category is denoted by 'CF'

Proof

Using Proposition 2.6 and Lemma 1.12(a) , it follows

Proposition 2.8

'Hom_R' is a invariant functor in the sense that it is both a covariant and a contravariant functor

Hom_R : CM→ CM is an invariant functor, for any fixed R-module M.

Proof:

Define Hom_R : CM→ CF by

Hom_R(N) = Hom_R(M,N), for any fixed R-module M, which is the object of CF.

Let N,P are two R-modules in CM and f:N→ P be R-homomorphisms in **C**M, then

Hom_R(f) = f_* : Hom_R(M,N) → Hom_R(M,P) in CF and f^* : Hom_R(P,M) → Hom_R(N,M) are well defined mapping and so by Definition 1.1, Lemma1.12 and Theorems 2.6, the theorem follows.

Proposition 2.9

'Hom_R' is a Homotopy type functor in the sense that if

f is a homotopy equivalence for any two R-modules M and N ,then Hom_R(f) is a isomorphism

Proof:

Since f is a homotopy equivalence for any two R-modules M and N, there exists f:M →N and g: N →M such that g.f $\simeq$ I_M and f. g $\simeq$ I_N, then Hom_R (f): Hom_R(P,M) →Hom_R(P,N) and Hom_R (f): Hom_R(N,P) →Hom_R(M,P) are fuzzy R- homomorphisms, then Hom_R satisfies the following conditions:

i) **f** $\simeq$ **g** ⇒ Hom_R(f) = Hom_R(g)

ii) g.f $\simeq$ I_M ⇒ Hom_R(g f) = Hom_R(I_M) = Id.⇒ Hom_R(g).Hom_R(f) = Id

iii) f. g $\simeq$ I_N ⇒ Hom_R(f g) = Hom_R(I_N) = Id.⇒ Hom_R(f).Hom_R(g) = Id

Thus Hom_R(f) is isomorphic to Hom_R(g)

Corollary 2.10

Hom_R is also a Homotopy type invariant functor.

Proposition 2.11

All homotopy type invariant functors form a function spaces from category **CM** to the category **CF,** it is denoted by F^M.

Proof:

Using the **Theorems 2.8**, **Theorem 2.9** and **Theorem 2.10**, it follows.

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