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Discovery

On the homogeneous cubic equation with four unknowns ($X^3+Y^3 = 14Z^3 - 3W^2(X+Y)$)

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ABSTRACT

The homogeneous cubic equation with four unknowns represented by the diophantine equation ($X^3+Y^3 = 14Z^3 - 3W^2(X+Y)$) is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special numbers are exhibited.

Keywords: Integral solutions, lattice points, homogeneous cubic equation with four unknowns.
M.Sc 2000 mathematics subject classification: 11D25

Notations: $t_{m,n}$: Polygonal number of rank n with size m, pr_n : Pronic number of rank n, J_n : Jacobsthal number of rank n, j_n : Jacobsthal - lucas number of rank n, ky_n : Kynea number of rank n

1. INTRODUCTION

The diophantine equations offer an unlimited field for research due to their variety (Dickson, 2005: Mordell, 1969; Carmichael, 1959). In particular, one may refer Gopalan et al. (2009, 2010a, 2010b, 2010c, 2010d, 2010e) for cubic equations with four unknowns. This communication concerns with yet another interesting equation ($X^3+Y^3 = 14Z^3 - 3W^2(X+Y)$) representing homogeneous cubic equation with four unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

2. METHOD OF ANALYSIS

The diophantine equation representing a homogeneous cubic equation with four unknowns is

$$x^3 + y^3 = 14z^3 - 3w^2(x+y) \quad (1)$$

To start with, it is observed that (1) is satisfied by the following integer quadruples (x, y, z, w)

i. $(2a^2 + b^2 + 2ab, 2ab - 2a^2 - b^2, 2ab, 2a^2 - b^2)$,

ii. $(4a^2 + 2b^2 + 6ab, -2ab, 2a^2 + b^2 + 2ab, 2a^2 - b^2)$,

iii. $(2\alpha^2 + 2\alpha\beta, 2\beta^2 - 2\alpha\beta, -\alpha^2 - \beta^2, -\alpha^2 + \beta^2 + 2\alpha\beta)$

iv. $(-2\beta^2 - 2\alpha\beta, -2\alpha^2 + 2\alpha\beta, -\alpha^2 - \beta^2, -\alpha^2 + \beta^2 - 2\alpha\beta)$

However, we have two more patterns of solutions for (1) which are illustrated below.

2.1. Pattern - I

Introducing the linear transformations

$$x = u + v, y = u - v, z = u \quad (2)$$

in (1), it leads to

$$w^2 + v^2 = 2u^2 \quad (3)$$

Again, applying the linear transformations

$$w = p - q, v = p + q \quad (4)$$

in (3),it leads to the Pythagorean equation $p^2 + q^2 = u^2$

which is satisfied by

$p = 2rs$

$q = r^2 - s^2$

$u = 9r^2 + s^2, r > s > 0$

Substituting the above values of p, q, u in (4) and (2), the corresponding non-zero distinct integral solutions of (1) are given by $x = x(r,s) = 2r^2 + 2rs$

$y = y(r,s) = 2s^2 - 2rs$

$z = z(r,s) = r^2 + s^2$

$w = w(r,s) = s^2 - r^2 + 2rs$

2.2. Properties

1. Each of the following is a nasty number.
   1. $6x(r,1)$, if $r = +\frac{1}{4}\{[\sqrt{2}+1]^{n+1} \mp [\sqrt{2}-1]^{n+1}\}^2$
   2. $6[z(2pq, p^2 - q^2)]$
   3. $3[x(2pq, p^2 - q^2) + y(2pq, p^2 - q^2)]$
   4. $6[x(r,1) + z(r,1) - 4t_{3,r} - 1]$
2. $w(r,s) + z(r,s) - y(r,s)$ is written as the difference of two squares
3. $x(2^n,1) + z(2^n,1) - 2(2^{2n} + 1)$ is kynea number
4. $x(2^n,1) + y(2^n,1) = j_{2n}$
5. $x(r,1) + y(r,1) + z(r,1) + w(r,1) - 2pr_r = 4$
6. $x(2^n,1) + y(2^n,1) - z(2^n,1) = j_{2n}$

2.3. Pattern - II

Assume

$$u = u(a,b) = (a^2 + b^2), \quad a,b \neq 0 \quad (5)$$

and write 2 as

$$2 = \frac{\left((1+i)(1-i)\right)^{2n+1}}{2^{2n}} \qquad (6)$$

Substituting (5) and (6) in (3) and employing the method of factorization, define

$$(w+iv) = \frac{(a+ib)^2(1+i)^{2n+1}}{2^n}, \qquad (7)$$

Equating real and imaginary parts in (7) we get,

$$w = w(n,a,b) = \sqrt{2}\left[(a^2-b^2)\cos\left(\frac{(2n+1)\pi}{4}\right) - 2ab\sin\left(\frac{(2n+1)\pi}{4}\right)\right]$$

$$v = v(n,a,b) = \sqrt{2}\left[2ab\cos\left(\frac{(2n+1)\pi}{4}\right) + (a^2-b^2)\sin\left(\frac{(2n+1)\pi}{4}\right)\right]$$

For simplicity and clear understanding, when $n = 0$, the corresponding solutions of (1) are given by

$$x = x(a,b) = 2a^2 + 2ab$$

$$y = y(a,b) = 2b^2 - 2ab$$

$$z = z(a,b) = a^2 + b^2$$

$$w = w(a,b) = a^2 - b^2 - 2ab$$

Note: It is worth to mention here that the above solution is same as in pattern I.

2.4. Properties

1. Each of the following is a nasty number.
   1 $6[z(2pq, p^2 - q^2)]$
   2 $6\left[x(a,a) + y(a,a) + z(a,a) + w(a,a)\right]$
   3 $2[x(a,1) + y(a,1) + z(a,1) - 3pr_3 - t_{10,a} - 7_{4,a} + 3]$
2. $x(2^n,1) + y(2^n,1) + z(2^n,1) = 3j_{2n}$
3. $x(1,b) + y(1,b) + z(1,b) - w(1,b) - 4t_{3,b} - 2pr_b \equiv 0 \pmod 2$
4. $z(a,1) + w(a,1) - pr_a - 2t_{3,a} \equiv 0 \pmod 2$
5. $x(2^n,1) + y(2^n,1) - 6J_{2n} = 4$

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