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Discovery

A unified study of inversion of an integral equation with the - function of two variables as its Kernel-II

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ABSTRACT
The object of this paper is to solve an integral equation of convolution from having the I -function of two variables as its kernel. Some special cases are also given in the end.

Keywords: Laplace Transform; Lerch's Theorem, I -function.
(2000 Mathematics Subject Classification: 33C99)

1. INTRODUCTION

1.1. Definition and Results

The Laplace Transform

$$F(p)=\int_0^\infty e^{-pt} f(t)dt,\ \mathrm{Re}(p)>0 \tag{1}$$

Is represented by $F(p)=L\{f(t)\}$.

Erdelyi (1954),

If $F(p)=L\{f(t)\}$ then

$$e^{-at}f(t)=F(p+a) \quad (2)$$

If $L\{f(t)\}=F(p),\ f(0)=f^{'}(0)=\ldots=f^{n-1}(0)=0$ and $f^{n}(t)$ is continuous then

$$L\{f^{n}(t)\}=p^{n}F(p) \quad (3)$$

If $f(t)$ is a continuous of t and $f(t)=L\{g(t)\}$, the integral $\int_0^{\infty} e^{-pt}t^{n}f(t)dt$ converges, then

$$t^{n}f(t)=\left(-\frac{d}{dp}\right)^{n}g(p) \quad (4)$$

If $L\{f(t)\}=F(p)$ and $L\{g(t)\}=G(p)$, then

$$\int_0^{\infty} f(x)g(p+x)dx=F(t)G(t) \quad (5)$$

The I-function introduced by Saxena (1982) will be represented and defined as follows:

$$I[Z]=I^{m,n}_{p_i,q_i:r}[Z]=I^{m,n}_{p_i,q_i:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L \chi(\xi)d\xi$$

(6)

where $\omega=\sqrt{-1}$

$$\chi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji})\prod_{j=n+1}^{p_i}\Gamma(a_{ji},\alpha_{ji})\right\}}$$

(7)

$p_i,q_i(i=1,\ldots,r),m,n$ are integers satisfying $0\le n\le p_i$, $0\le m\le q_i,(i=1,\ldots,r),r$ is finite $\alpha_j,\beta_j,\alpha_{ij},\beta_{ji}$ are real and a_j,b_j,a_{ji},b_{ji} are complex numbers such that

$\alpha_j(b_h+v)\ne\beta_h(a_j-v-k)$ for $v,k=01,2,\ldots$

We shall use the following notations:

$A^{*}=(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i};B^{*}=(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}$

The I-function of two variables introduced by Prasad (1986) will be represented and defined as follows:

$$I[z_1,z_2]=I^{0,n_2:(m'.n'):(m'',n'')}_{p_2,q_2:(p',q'):(p'',q'')}\left[\begin{matrix}z_1\\z_2\end{matrix}\left|\begin{matrix}(a_{2j}:\alpha'_{2j},\alpha''_{2j})_{1,p_2}:(a'_j,\alpha'_j)_{1,p'}:(a'_j,\alpha'_j)_{1,p''}\\(b_{2j}:\beta'_{2j},\beta''_{2j})_{1,q_2}:(b'_j,\beta'_j)_{1,q'}:(b'_j,\beta'_j)_{1,q''}\end{matrix}\right.\right]$$

$$=\frac{1}{(2\pi w)^2}\int_{L_1}\int_{L_2}\phi_1(s_1)\phi_2(s_2)\psi(s_1,s_2)z_1^{s_1}z_2^{s_2}ds_1ds_2 \quad (8)$$

Where $w=\sqrt{-1}$,

$$\phi_i(s_i)=\frac{\prod_{j=1}^{m^{(i)}}\Gamma\left(b_j^{(i)}-\beta_j^{(i)}s_i\right)\prod_{j=1}^{n^{(i)}}\Gamma\left(1-a_j^{(i)}+\alpha_j^{(i)}s_i\right)}{\prod_{j=m^{(i)}+1}^{q^{(i)}}\Gamma\left(1-b_j^{(i)}+\beta_j^{(i)}s_i\right)\prod_{j=n^{(i)}1}^{p^{(i)}}\Gamma\left(a_j^{(i)}-\alpha_j^{(i)}s_i\right)}\quad\forall i\in\{1,2\}$$

(9)

$$\psi(s_1,s_2)=\frac{\prod_{j=1}^{n_2}\Gamma\left(1-a_{2j}+\sum_{i=1}^{2}a_{2j}^{(i)}s_i\right)}{\prod_{j=n_2+1}^{p_2}\Gamma\left(a_{2j}-\sum_{i=1}^{2}a_{2j}^{(i)}s_i\right)\prod_{j=1}^{q_2}\Gamma\left(1-b_{2j}+\sum_{i=1}^{2}\beta_{2j}^{i}s_i\right)}$$

(10)

We will use the following result

$$\Gamma(v_1)(p+a)^{-1-h_1}\left[1+z_1(p+a)p^{k_1}\right]^{-v_1}\Gamma(v_2)(p+a+b)^{-1-h_2}\left[1+z_2(p+a+b)^{k_2}\right]^{-v_2}$$

$$=\sum_{r=0}^{\infty}\frac{b^r}{r!}e^{-(b+a)t}t^{r+h_1+h_2+1}I^{0,0:2,1:1,1}_{1,0:2,1:1,1}\left[\begin{matrix}z_1t^{-k_1}\\z_2t^{-k_2}\end{matrix}\left|\begin{matrix}(r+h_1+h_2+2,k_1,k_2),(1-v_1,k_1),(1+h_1,k_1),(1-v_2,1)\\(\ldots:(r+h_1+1,k_1),(0,1),(0,1)\end{matrix}\right.\right]$$

(11)

Provided

$$\mathrm{Re}(1+h'_1+k_1v_1)>0,\mathrm{Re}(1+h'_2+k_2v_2)>0,|\arg z_1p^{k_1}|<\frac{\pi}{2}(2-k_1),$$

$$|\arg z_2p^{k_2}|<\frac{\pi}{2}(2-k_2),2>k_1>0,2>k_2>0,\mathrm{Re}(p+a)>0,\mathrm{Re}(p+a+b)>0$$

2. MAIN RESULT

Theorem: Each of the integral equations

$$G(p)=A\sum_{r=0}^{\infty}\frac{b^r}{r!}\int_0^{\infty}\left[(a-D)^{m_1}(a+b-D)^{m_2}F(p+x)\right]e^{-(a+b)x}x^{r+h_1+h_2+1}$$

$$\times I^{0,0:2,1:1,1}_{1,0:2,2:1,1}\left[\begin{matrix}z_1x^{-k_1}\\z_2x^{-k_2}\end{matrix}\left|\begin{matrix}(r+h_1+h_2+2,k_1,k_2):(1-v_1,1),(1+h_1,k_1):(1-v_2,1)\\\ldots:(r+h_1+1,k_1),(0,1),(0,1)\end{matrix}\right.\right]dx$$

(12)

And

$$F(p)=B\sum_{r=0}^{\infty}\frac{b^r}{r!}\int_0^{\infty}\left[(a-D)^{n_1}(a+b-D)^{n_2}G(p+x)\right]e^{-(a+b)x}x^{r+h'_1+h'_2+1}$$

$$\times I^{0,0:2,1:1,1}_{1,0:2,2:1,1}\left[\begin{matrix}z_1x^{-k_1}\\z_2x^{-k_2}\end{matrix}\left|\begin{matrix}(r+h'_1+h'_2+2,k_1,k_2):(1-v_1,1),(1+h'_1,k_1):(1-v_2,1)\\\ldots:(r+h'_1+1,k_1),(0,1),(0,1)\end{matrix}\right.\right]dx$$

(13)

In the solution of the other, provided

$$m_1+n_1=h_1+h'_1+2,m_2+n_2=h_2+h'_2+2$$

$$AB\Gamma(v_1)\Gamma(v_2)\Gamma(-v_1)\Gamma(-v_2)=1,\mathrm{Re}(p)>0,$$

$$\mathrm{Re}(1+h_1+k_1v_1)>0,\mathrm{Re}(1+h_2+k_2v_2)>0,$$

$$\mathrm{Re}(1+h'_1-k_1v_1)>0,\mathrm{Re}(1+h'_2-k_2v_2)>0,|\arg z_1p^{k_1}|<\frac{\pi}{2}(2-k_1),$$

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$$|\arg z_2 p^{k_2}| < \frac{\pi}{2}(2-k_2), 2>k_1>0, 2>k_2>0, \mathrm{Re}(p+a)>0, \mathrm{Re}(p+a+b)>0$$

m_1, m_2, n_1 and n_2 are integers.

D represents differentiation with respect to $(p+x)$.

2.1. Proof

Let $L\{f(t)\} = F(p)$ and $L\{g(t)\} = G(p)$

$$(a-D)^{m_1}(a+b+D)^{m_2} F(p) = (a+t)^{m_1}(a+b+t)^{m_2} f(t) \tag{14}$$

$$(a-D)^{n_1}(a+b+D)^{n_2} G(p) = (a+t)^{n_1}(a+b+t)^{n_2} g(t) \tag{15}$$

With the help of (5) and (11), the integral equation (12) gives

$$g(t) = A\Gamma(v_1)(t+a)^{m_1-1-h_1}(t+a+b)^{m_2-1-h_2} f(t)\left[1+z_1(t+a)p^{k_1}\right]^{-v_1} \Gamma(v_2)\left[1+z_2(t+a+b)^{k_2}\right]^{-v_2}$$

16)

Similarly, the integral equation (13) gives

$$f(t) = B\Gamma(-v_1)(t+a)^{n_1-1-h_1}(t+a+b)^{n_2-1-h_2} g(t)\left[1+z_1(t+a)p^{k_1}\right]^{-v_1} \Gamma(-v_2)\left[1+z_2(t+a+b)^{k_2}\right]^{-v_2}$$

(17)

The equations (16) and (17) can be obtained from each other when

$$AB\Gamma(v_1)\Gamma(v_2)\Gamma(-v_1)\Gamma(-v_2) = 1, \mathrm{Re}(p)>0, 2>k_1>0, 2>k_2>0$$

$m_1+n_1 = h_1+h'_1+2$ and $m_2+n_2 = h_2+h'_2+2$

Hence by Lerch's theorem ((1962), p.5), it follows that each of the integral equations (12) and (13) is the solution of the other.

3. SPECIAL CASES

In the theorem put $k_2 = 1, k_1 = k, v_1 = v, z_1 = z$ and make $z_2 \to 0$ to get the following result involving I-function of one variable.

Each of the integral equations

$$G(p) = A\sum_{r=0}^{\infty}\frac{b^r}{r!}\int_0^{\infty}\left[(a-D)^{m_1}(a+b-D)^{m_2} F(p+x)\right] e^{-(a+b)x} x^{r+h_1+h_2+1} \times I_{3,2}^{2,1}\left[zx^{-k}\left|\begin{matrix}(r+h_1+h_2+2,k):(1-v,1),(1+h_1,k)\\ \ldots:(r+h_1+1,k),(0,1)\end{matrix}\right.\right]dx \tag{18}$$

And

$$F(p) = B\sum_{r=0}^{\infty}\frac{b^r}{r!}\int_0^{\infty}\left[(a-D)^{n_1}(a+b-D)^{n_2} G(p+x)\right] e^{-(a+b)x} x^{r+h'_1+h'_2+1} \times I_{3,2}^{2,1}\left[zx^{-k}\left|\begin{matrix}(r+h'_1+h'_2+2,k):(1+v,1),(1+h'_1,k)\\ \ldots:(r+h'_1+1,k),(0,1)\end{matrix}\right.\right]dx \tag{19}$$

In the solution of the other, provided the conditions of Theorem are satisfied with

$AB\Gamma(v)\Gamma(-v) = 1,$ and $2>k>0$

When

$h_1 = \alpha, h'_1 = \beta, h_2 = h'_2 = -1, m_1 = m, n_1 = n, m_2 = n_2 = 0$

and $b \to 0$, (18) and (19) reduces to:

Each of the integral equations

$$G(p) = A\int_0^{\infty}\left[(a-D)^m f(p+x)\right] e^{-ax} x^{\alpha} \times I_{2,1}^{1,1}\left[zx^{-k}\left|\begin{matrix}(1+\alpha,k):(1-v,1)\\ (0,1)\end{matrix}\right.\right]dx \tag{20}$$

And

$$F(p) = B\int_0^{t}\left[(a-D)^n G(p+x)\right] e^{-ax} x^{\beta} \times I_{2,1}^{1,1}\left[zx^{-k}\left|\begin{matrix}(1+\beta,k):(1+v,1)\\ (0,1)\end{matrix}\right.\right]dx \tag{21}$$

is the solution of the other, provided

m and n are integers, $m+n = 2+\alpha+\beta$

$f(0) = f'(0) = \ldots = f^{m-1}(0) = 0, and\ f^m(t)$ is continuous when $m>0$

$g(0) = g'(0) = \ldots = g^{n-1}(0) = 0, and\ g^n(t)$ is continuous when $n>0$

$\mathrm{Re}(1+\alpha+kv)>0, 2>k>0, \mathrm{Re}(1+\beta-kv)>0$.

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